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Extending an Order on a Set to the Power Set: Some Remarks on Kannai and Peleg's Approach*

SALVADOR BARBERÁ

Departamento de Teoría Económica, Facultad de Ciencias Económicas, Universidad del País Vasco, Av. Ejército 87, Bilbao-14, Spain

AND

PRASANTA K. PATTANAIK

Department of Economics, University of Birmingham, Birmingham B15 2TT, England

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Kannai and Peleg have shown that given an ordering over a set, it is impossible to induce an ordering over the power set satisfying certain plausible axioms. We prove an impossibility and also a possibility result in this context with closely related sets of axioms, and argue that the dividing line between impossibility and possibility here is rather thin. Also, we distinguish three possible intuitive interpretations for the formal framework of Kannai and Peleg, and argue that the acceptability of specific formal axioms may crucialy depend on the particular interpretation that one chooses to adopt. *Journal of Economic Literature* Classification Numbers: 025, 026.

1. INTRODUCTION

In an important and elegant paper, Kannai and Peleg [4] have recently proved that an order on a set cannot be extended to the power set while satisfying two prima facie attractive axioms. This note contains two propositions in the same vein, one being positive and the other negative in their conclusions. The main purpose of the note is, however, not so much to prove these results per se but rather to use them as accompanying illustrations for our comments on Kannai and Peleg's contribution. To a certain extent, our propositions show how very similar looking sets of

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axioms may lie on opposite sides of the possibility–impossibility frontier. This leads us to make the point that Kannai and Peleg's very well placed emphasis on trade-offs between axioms should be complemented by appropriate semantic interpretations of the extended relation one is looking for, under which different axioms and their trade-offs convey more specific meaning.

2. The Elements of the Problem

Following Kannai and Peleg, let Ω be a set and let R be a fixed linear ordering over Ω. Let $\succcurlyeq$ denote a reflexive, binary relation on 2^{Ω}, the set of non-empty subsets of Ω. For all $A, B \in 2^{\Omega}$, $A \succ B$ iff $[A \succcurlyeq B$ and not $B \succcurlyeq A]$; and $A \sim B$ iff $[A \succcurlyeq B$ and $B \succcurlyeq A]$. Where A is any finite subset of Ω, we denote by $\max(A)$ (resp. $\min(A)$) the greatest (resp. smallest) member of A in the order R.

Conditions can be placed on the relationship between R and $\succcurlyeq$, under which $\succcurlyeq$ could be regarded as a "satisfactory" extension over sets of alternatives of the relation R on single alternatives. Kannai and Peleg propose two such conditions in the form of axioms, labelled (GP) (the Gardenförs principle) and (M) (monotonicity).

(GP) Let A be a finite subset of Ω and $x \in \Omega - A$. If $xR \max(A)$, then $A \cup \{x\} \succ A$, and if $\min(A) Rx$, then $A \succ A \cup \{x\}$.

(M) If A, B, $C \in 2^{\Omega}$, $A \cap B = A \cap C = \varnothing$ and $B \succ C$, then $A \cup B \succcurlyeq A \cup C$.

We want to consider the following related axioms.

(B) For all distinct $x, y \in \Omega$, if xRy, then $\{x\} \succ \{x, y\}$ and $\{x, y\} \succ \{y\}$.

(K) Let $A, B \in 2^{\Omega}$. If (for all $x \in A$ and all $y \in B$, xRy) and (for some $x \in A$ and some $y \in B$, xRy and not yRx), then $A \succ B$.

Since R is a strict ordering, (K) can be seen to be equivalent to

(K′) For all distinct $A, B \in 2^{\Omega}$, if for all $x \in A$ and all $y \in B$, xRy, then $A \succ B$.

Clearly, (GP) $\rightarrow$ (K) $\rightarrow$ (B), and the converse implications do not hold. Axioms (B) and (K) were introduced by Barberá [1] and Kelly [5] in their treatments of strategy-proof set-valued social choice mechanisms.

Consider also the following monotonicity axiom, (M′).

(M′) If $A, B, C \in 2^{\Omega}$, $A \cap B = A \cap C = \varnothing$, and $B \succ C$, then $A \cup B \succ A \cup C$.

(M′) is somewhat stronger than (M) but under many important intuitive interpretations of our formal problem it is difficult to see why one would object to (M′) while accepting (M). (We return to this problem in Section 4, where we take up the problem of intuitive interpretation.) If $\succcurlyeq$ is connected over 2^{Ω} (i.e., if for all distinct $A, B \in 2^{\Omega}$, $A \succcurlyeq B$ or $B \succcurlyeq A$) then (M′) is equivalent to the following axiom (see Krantz *et al.* [6, p. 204]).

(M″) If $A, B, C \in 2^{\Omega}$, and $A \cap B = A \cap C = \varnothing$, then $B \succcurlyeq C$ iff $A \cup B \succcurlyeq A \cup C$.

3. The Results

Kannai and Peleg prove the following.

Proposition 1. *If Ω contains at least five members, then there exists no weak order $\succcurlyeq$ on 2^{Ω} which satisfies* (GP) *and* (M).

Our first result shows that relaxing (GP) to (K) is sufficient to turn the above impossibility into a possibility result.[1] Since our purpose is essentially conceptual rather than technical we restrict ourselves to the case where Ω is finite.

Proposition 2. *If Ω is finite there exists a weak order over 2^{Ω} satisfying* (K) and (M).

Proof. For all $E \in 2^{\Omega}$ let $\underset{\sim}{E} = E - \{\max(E)\}$. Define $\succcurlyeq$ over 2^{Ω} as follows: for all $A, B \in 2^{\Omega}$, $A \succcurlyeq B$ iff

(1) not $[\max(B)\, R \max(A)$ & $\max(B) \neq \max(A)]$; and (2) not $[\max(B) = \max(A)$ & $\underset{\sim}{B} = \varnothing$ & $\underset{\sim}{A} \neq \varnothing]$.

$\succcurlyeq$ is clearly reflexive and connected. To show transitivity of $\succcurlyeq$ assume that $A \succcurlyeq B$ & $B \succcurlyeq C$ for some $A, B, C \in 2^{\Omega}$. Then using the definition of $\succcurlyeq$, and noting that R is a strict ordering,

either (3) $\max(A)\, R \max(C)$ & $\max(A) \neq \max(C)$
or (4) $[\max(A) = \max(B) = \max(C)]$ & $([\underset{\sim}{B} = \varnothing$ & $\underset{\sim}{A} = \varnothing]$ or $[\underset{\sim}{C} \neq \varnothing$ & $\underset{\sim}{B} \neq \varnothing])$.

If (3) holds, $A \succ C$. If (4) holds, then $[\max(A) = \max(C)$ & $(\underset{\sim}{A} = \varnothing$ or $\underset{\sim}{C} \neq \varnothing)]$ in which case $A \succcurlyeq C$. Then in all cases $A \succcurlyeq C$ and the proof of transitivity of $\succcurlyeq$ is complete.

[1] For an alternative possibility result obtained through a different modification of Kannai–Peleg axioms, see Fishburn [2].

We show that the weak order $\succcurlyeq$ satisfies (K) and (M). First consider (K). Noting that (K) is equivalent to (K′), consider distinct $D, E \in 2^{\Omega}$ such that

(5) for all $x \in D$ and all $y \in E$, xRy.

Clearly, $\max(D)\, R \max(E)$. If $\max(D) \neq \max(E)$, it would follow that $D \succ E$. Suppose $\max(D) = \max(E)$. If $\underset{\sim}{D} \neq \varnothing$, then we would have $xR \max(E)$ for all $x \in \underset{\sim}{D}$. Given $\max(D) = \max(E)$, this cannot be true. Hence, given $\max(D) = \max(E)$, $\underset{\sim}{D} = \varnothing$. Since $[\underset{\sim}{D} = \varnothing$ & $\max(D) = \max(E)$ & $D \neq E]$, $\underset{\sim}{E} \neq \varnothing$. It follows that $D \succ E$. Thus in all cases $D \succ E$, and therefore $\succcurlyeq$ satisfies (K).

Now consider (M). Suppose for some A, B, $C \in 2^{\Omega}$, $B \succ C$ and $B \cap A = C \cap A = \varnothing$. Since $B \succ C$,

either (6) $[\max(B)\, R \max(C)$ & $\max(B) \neq \max(C)]$,
or (7) $[\max(B) = \max(C)$ & $\underset{\sim}{C} \neq \varnothing$ & $\underset{\sim}{B} = \varnothing]$.

Irrespective of whether (6) holds or (7) holds, $\max(B)R \max(C)$. Hence $\max(A \cup B)\, R \max(A \cup C)$. If $\max(A \cup B) \neq \max(A \cup C)$, then $A \cup B \succ A \cup C$. Suppose $\max(A \cup B) = \max(A \cup C)$. Since $\underset{\sim}{C} \neq \varnothing$, indicating $A \cup C$ by H, we must have $\underset{\sim}{H} \neq \varnothing$. Then it follows that $A \cup B \succcurlyeq A \cup C$. This completes the proof. ∎

The possibility result in Proposition 2 is rather tenuous. The following proposition shows that if we replace (M) by the very similar but formally stronger (M′), then it is impossible to have any binary relation $\succcurlyeq$ over 2^{Ω} satisfying (M′) and meeting even the minimal axiom (B), which is strictly implied by (K) and, a fortiori, by (GP).

Proposition 3. *If Ω contains at least three members, there exists no binary relation $\succcurlyeq$ on 2^{Ω} which satisfies* (B) *and* (M′).

Proof. Suppose there exists a binary relation over 2^{Ω} satisfying the two conditions. Let x_1, x_2, x_3 be three distinct elements of Ω such that $x_1 R x_2 R x_3$. Then, by (B), we have (i) $\{x_1\} \succ \{x_1, x_2\}$, and (ii) $\{x_2, x_3\} \succ \{x_3\}$. Then, by (M′), from (i) we have $\{x_1, x_3\} \succ \{x_1, x_2, x_3\}$, and by (M′) again, from (ii), we have $\{x_1, x_2, x_3\} \succ \{x_1, x_3\}$. This gives us a contradiction which completes the proof. ∎

Note that in Proposition 3 we do not assume the binary relation $\succcurlyeq$ to be a weak ordering. The proposition holds even if $\succcurlyeq$ does not satisfy connectedness or transitivity. This contrasts with the result of Kannai and Peleg, who assume $\succcurlyeq$ to be an ordering and rely heavily on transitivity in their proof.

4. Comments

Our main purpose with these two results was to show how thin is the frontier between possibility and impossibility results in this context, and how hard it is to evaluate any of them without some further reference to specific interpretations. Obviously it makes a big difference whether we accept axiom (M) or (M′); although neither of the two is compatible with (GP) and the requirement that $\succcurlyeq$ should be a weak order, (M) is compatible with (K) (a very reasonable weakening of (GP)) and the requirement that $\succcurlyeq$ should be a weak order, while even in the absence of any further assumption regarding the reflexive and binary relation $\succcurlyeq$, (M′) is incompatible with (B) which is the mildest requirement one can think of within its class. Is there any reason to think that (M′) is unacceptable while (M) is acceptable? There does not seem to be any such convincing reason under what we find to be the most interesting interpretation of the formal framework outlined in the earlier sections. This we discuss below. However, the point that we want to emphasize here is that without a specific intuitive interpretation it is practically impossible to discuss questions about the acceptability of axioms such as (M′), (M), or (B). It may be useful to illustrate this with three alternative interpretations of our formal problem.

(1) First consider the case where the agent may be presented with any one of several feasible sets of alternatives. Given a feasible set, the agent chooses an alternative from the feasible set (and gets the alternative thus chosen). Here $\succcurlyeq$ is to be interpreted as reflecting the relative desirability for the agent, of the various possible feasible sets. In this context (M′) runs into intuitive problems. Suppose for some distinct alternatives x, y, and z, $xRyRz$ (note that R is a linear ordering), and suppose $A = \{x\}$, $B = \{y\}$, and $C = \{z\}$. It is plausible to assume $B \succ C$ but there is no reason to expect $A \cup B \succ A \cup C$. This is because the best alternative of $A \cup B$ and also of $A \cup C$ lies in A and is the same in both cases. Since the agent will choose the best alternative in each case, there is no reason to consider the feasible set $A \cup B$ more desirable then the feasible set $A \cup C$. Thus (M′) creates intuitive problems in this context, which do not arise in the case of (M). However, note that here the difficulty is not confined to (M′) alone. Even (B) is subject to the same difficulties (given $x \neq y$ and xRy, $\{x\}$ and $\{x, y\}$ will be equaly desirable under this interpretation). Also note that under this interpretation, the problem of generating $\succcurlyeq$ over 2^{Ω}, from R over Ω, becomes somewhat trivial since intuitively the relative desirability of any two feasible sets will be exclusively determined by the relative desirability of their best elements (so that the simple "maxi-max" rule will seem to be the only acceptable way of ranking feasible sets).

(2) Consider now a second interpretation of the problem of inducing $\succcurlyeq$

over 2^{Ω} given an ordering R over Ω. This is the interpretation which we find most interesting and which provided our main motivation for studying the formal problem. Here the agent chooses one of several alternative actions. Corresponding to each action a, there is a set $F(a) \in 2^{\Omega}$ of possible, mutually exclusive outcomes, but the agent does not have probabilities for these outcomes (thus the choice problem is characterized by "uncertainty" rather than "risk").[2] Given the ordering R over the set of all conceivable outcomes Ω, the agent tries to rank actions which amounts to ranking elements of 2^{Ω}. Under this interpretation it is no longer clear why one would intuitively accept (M) but not (M′). The objection against (M′) given above does not apply here.

(3) Our third interpretation is in terms of the problem of plausible reasoning discussed by various writers (see, for example, Rescher [10] and Packard [8]). Here we have an inconsistent set Ω of statements over which we have an ordering R in terms of plausibility. The objective is to rank—in terms of plausibility again—the various consistent subsets of Ω. This is formally exactly similar to the second problem described above. However, it is not at all clear that formally identical axioms will have the same intuitive appeal in both cases especially since in the problem of choice under uncertainty, the elements of Ω are to be interpreted as mutually exclusive outcomes, while in the problem of plausible reasoning the statements in Ω are not necessarily pairwise inconsistent even though Ω is an inconsistent set of statements.

Work on the connection between preferences on singletons and preferences on the power set, prior to Kannai and Peleg, had taken rather a casuistic view: specific extensions had been proposed, or specific criteria leading to such extensions.[3] The motivation underlying such proposals is generally not explicit, although one can usually think of circumstances and interpretations under which a specific extension would be reasonable. By adopting an axiomatic treatment, Kannai and Peleg rightly place the problems of extension in an "economic" setting, where no specific criterion is advanced, and the emphasis is on the trade-offs between alternative types of properties that an extension may or may not satisfy. The natural questions within this framework involve the existence and the characterization of extensions satisfying collections of such requirements, and both possibility and impossibility results may arise, depending on the number and the strength of the required properties. The end result of a thorough investigation would map a possibility set, and most importantly a possibility frontier, in the "space" of desirable properties of extensions. However, it seems to us—and this is what we have tried to argue above—that it may not be possible to

[2] Cf. Luce and Raiffa [7].

[3] See, among others, Barberá [1], Gardenförs [3], Kelly [5] and Pattanaik [9].

define clearly the "coordinate axes" of that "space" of desirable properties unless one attaches a specific interpretation to our sought extension. Without such interpretation, any proposed property would seem a trifle ad hoc and its interest would lie mainly in the unexplicated common background and interpretation of a group of researchers.

REFERENCES

1. S. BARBERÁ, The manipulability of social mechanisms that do not leave too much to chance, *Econometrica* **45** (1977), 1573–1588.
2. P. FISHBURN, Comment on the Kannai–Peleg impossibility theorem for extending orders, *J. Econ. Theory* **32** (1984), 176–179.
3. P. GARDENFÖRS, Manipulation of social choice functions, *J. Econ. Theory* (1976), 217–228.
4. Y. KANNAI AND B. PELEG, A note on the extension of an order on a set to the power set, *J. Econ. Theory* **32** (1984), 172–175.
5. J. KELLY, Strategy proofness and social choice functions without singlevaluedness, *Econometrica* **45** (1977), 439–446.
6. D. KRANTZ, R. LUCE, P. SUPPES, AND A. TVERSKY, "Foundation of Measurement," Wiley, New York, 1971.
7. R. D. LUCE AND H. RAIFFA, "Games and Decisions," Wiley, New York, 1957.
8. D. PACKARD, Plausibility orderings and social choice, *Synthese* **49** (1981), 415–418.
9. P. K. PATTANAIK, "Strategy and Group Choice," North-Holland, Amsterdam, 1978.
10. N. RESCHER, "Plausible Reasoning," Van Gorgum, Assem/Amsterdam, 1976.

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